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Phases of supersymmetric gauge theories from M-theory on G_2 manifolds

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Abstract: We consider M-theory on compact spaces of G_2 holonomy constructed as orbifolds of the form $(\mathrm{CY}\times S^1)/\mathbb{Z}_2$ with fixed point set Σ on the CY. This describes $\mathcal{N}=1$ SU(2) gauge theories with $b_1(\Sigma)$ chiral multiplets in the adjoint. For $b_1=0$, it generalizes to compact manifolds the study of the phase transition from the non-abelian to the confining phase through geometrical S^3 flops. For $b_1=1$, the non-abelian and Coulomb phases are realized, where the latter arises by desingularization of the fixed point set, while an $S^2\times S^1$ flop occurs. In addition, an extremal transition between G_2 spaces can take place at conifold points of the CY moduli space where unoriented membranes wrapped on $\mathbb{CP}^1$ and $\mathbb{RP}^2$ become massless.

Keywords: M-Theory, Gauge Symmetry.

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Contents

1. Introduction

There are many ways to obtain $\mathcal{N} = 1$ theories in 4D. The initial example is the heterotic compactification on Calabi-Yau threefolds. The SU(3) holonomy of the later allows a surviving Killing spinor and leads to an $\mathcal{N} = 1$ gauge theory in 4D. Similarly, a compactification of M-theory on a 7-manifold of G_2-holonomy gives rise to an $\mathcal{N} = 1$ theory in 4D. However, since there are no gauge fields to start with in 11D, the vector multiplets are generically abelian and thus of restricted interest. Their number depends on the topological properties of the internal space and equals the number of non-contractible two-cycles. On the other hand, it is known that the appearance of singularities in the internal space leads to non-abelian gauge theories. As has been shown in [1], in certain cases, as two-cycles are collapsing, A_n singularities appear and a corresponding enhancement of the gauge group occurs. This has explicitly been shown for the KK monopole when the ALE space degenerates to $\mathbb{R}^4/\mathbb{Z}_n$. In this limit, there exists an enhancement of the gauge group to $\mathrm{SU}(n)$ due to membranes wrapped on intersecting two-cycles. This is the way non-abelian vectors can appear in M-theory. This mechanism was also considered in [2] in order to study $\mathcal{N} = 1$ 6D theories at the conformal point in the AdS/CFT correspondence.

M-theory compactifications on G_2-holonomy manifolds have appeared in the past [3]. Recently, phase transitions between different G_2 manifolds have been considered in [4] that describe changes of branches in the scalar potential or a Higgs mechanism in the abelian sector of the theory. Moreover, the possibility of understanding the non-abelian phase structure of M-theory on G_2 manifolds was pointed out in [5, 6, 7]. The central ingredient in [6] is an S^3-flop in the underlying geometry. The flop has been interpreted as a phase transition of the gauge theory. On one side the SU(N) gauge group is in its non-abelian phase, while on the other side it has disappeared as an effect of confinement. However, the G_2 space which has been employed in this construction was non-compact.

In the present work we describe these effects in the case of compact spaces of G_2-holonomy constructed as orbifolds of the form $(\mathrm{CY}\times S^1)/\mathbb{Z}_2$. The $\mathbb{Z}_2$ acts as an inversion on the S^1 coordinate x^{10} and antiholomorphically on the CY, so that $J \to -J$ and $\Omega \to \bar{\Omega}$, where J, Ω are the Kähler form and the holomorphic 3-form. As a result,

$$\Phi = J \wedge dx^{10} + \mathrm{Re}(\Omega) \tag{1.1}$$

provides the orbifold with a G_2 structure [8]. The fixed point set Σ of the antiholomorphic involution on the CY are special lagrangian 3-cycles. When it is composed of P disconnected components Σ_p ($p = 1, \dots, P$), each of them is promoted to an associative 3-cycle of A_1 singularities on the 7-space. In M-theory compactifications, the first Betti number $b_1(\Sigma_p)$ of this cycle then counts the number of chiral multiplets in the adjoint representation of an SU(2) gauge group. The case $b_1(\Sigma_p) = 0$, to be considered in section 2 generalizes the work of [6] to a SU(2) gauge group on compact G_2 spaces. When $b_1(\Sigma_p) = 1$, we will see in section 3 that the non-abelian and Coulomb phases also occur in M-theory. The Coulomb phase will be described geometrically as the desingularization of the orbifold fixed points. In addition, $S^2 \times S^1$ flops will take place.

From the non-abelian point of view, the transitions in the two models with $b_1(\Sigma) = 0$ that we will present, occur at different types of singular points in the CY complex structure moduli space. In one case the CY has ordinary nodes and a conifold transition [9] gives rise to an additional descendant G_2 branch. This will generalize the other non-compact G_2 manifold with A_1 singularities appearing in the literature [4]. These transitions are triggered by black holes [10, 11] described as *unoriented* membranes wrapped on non-calibrated $\mathbb{CP}^1$ and $\mathbb{RP}^2$. A discussion along these lines from the supergravity point of view can be found in [12].

We remark that these models are dual to type-IIA compactified on the CY taking part in the orbifold in the presence of $2b_1(\Sigma_p)$ D6-branes and $b_1(\Sigma_p)$ O6-planes wrapped on Σ_p [13]–[17].

2. Models without adjoint matter

Our aim is to construct G_2 manifolds out of products $\mathrm{CY}\times S^1$ on which we act by antiholomorphic involutions w on the CY and by inversion $\mathcal{I}$ on S^1. In [4], freely acting involutions w on the whole CY ambient space were considered that result in smooth 7-dimensional orbifolds. Upon compactifying M-theory on them, transitions between topologically different such manifolds were used to describe transitions between distinct branches in the scalar potential of the associated $\mathcal{N}=1$ abelian effective field theory. However, it was noticed that if the involution has fixed points in the CY ambient space, a subset of them can live on the 7-dimensional G_2-orbifold and may or may not be desingularized while keeping the holonomy within G_2. The fixed point set on the CY is then a special lagrangian 3-cycle whose topology can vary when changing the CY complex structure. As a result it was claimed that a single CY moduli space can be split into distinct components describing G_2 orbifolds with different fixed point set topologies. When these orbifolds can be desingularized, they give manifolds with different Betti numbers due to different "twisted sectors" as will be seen in section 3. In M-theory compactifications they give the various phases of the $\mathcal{N}=1$ non-abelian effective field theory.

For the moment, we would like to consider the simplest case, where a transition takes place between a phase without fixed points to a phase with fixed point topology of disjoint 3-spheres. This transition was actually already considered in [6] and [4] in a local version on non-compact G_2 manifolds, namely, the spin bundle over S^3 and $(T^*S^3\times\mathbb{R})/\mathbb{Z}_2$, respectively. They both describe phase transitions in pure SU(2) gauge theory. However, the model in [4] has an additional G_2 branch arising from a conifold transition of the underlying CY T^*S^3. These effects are not restricted to non-compact G_2 spaces and this section is dedicated to demonstrate them in a compact model. In order to keep the discussion as simple as possible, we consider G_2 manifolds based on CY hypersurfaces in (the resolution of) the well-known weighted projective space $\mathbb{CP}^4_{11222}$.

2.1 A model based on $(\mathbb{CP}^4_{11222}[8]\times S^1)/\mathbb{Z}_2$

We define the one-parameter sub-set of threefolds $\mathcal{C}_1$ in the family $\mathbb{CP}^4_{11222}[8]$ with Hodge numbers $h_{11}=2$, $h_{12}=86$ by

$$p_1 = z_6^4(z_1^8+z_2^8-2\phi z_1^4z_2^4)+z_3^4+z_4^4+z_5^4=0\,, \tag{2.1}$$

where the projective coordinates are subject to the identifications under the two $\mathbb{C}^*$ actions

$$\begin{array}{c|cccccc} & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ \hline \mathbb{C}^*_1 & 0 & 0 & 1 & 1 & 1 & 1 \\ \mathbb{C}^*_2 & 1 & 1 & 0 & 0 & 0 & -2 \end{array}\,. \tag{2.2}$$

The additional variable z_6 accompanied by a second scaling action arises from blowing up the $\mathbb{Z}_2$ singularity $\{z_1=z_2=0\}$ of $\mathbb{CP}^4_{11222}$ which the CY family intersects. The resulting toric ambient space has an excluded set

$$(z_1,z_2)\neq(0,0)\quad\text{and}\quad(z_3,z_4,z_5,z_6)\neq(0,0,0,0)\,. \tag{2.3}$$

Notice that among the large number of monomials allowed by the $\mathbb{C}^*$ actions, we choose to consider only one, namely $z_1^4 z_2^4 z_6^4$, in eq. (2.1) with arbitrary complex coefficient.

A CY in this family happens to be singular when the equation (2.1) is non-transverse, i.e. when $p = \partial_{z_i} p = 0$, $(i = 1, \dots, 6)$. This occurs only when $\phi = \pm 1$ and for $\{z_1^4 = \phi z_2^4\} \cap \{z_3 = z_4 = z_5 = 0\}$. Thanks to eq. (2.3) these singularities lie in charts where z_2 and z_6 are non vanishing, so that we can rescale them to unity. In each case, there are thus 4 singular points,

$$\begin{array}{ll} \text{for } \phi = +1 : & (i^k, 1, 0, 0, 0, 1) \qquad\quad , (k = 0, \dots, 3) \\ \text{for } \phi = -1 : & (i^k e^{i\pi/4}, 1, 0, 0, 0, 1)\ , (k = 0, \dots, 3)\ . \end{array} \tag{2.4}$$

At these points, the determinant of second derivatives $\det(\partial_A \partial_B p_1)$ $(A, B = 1, 3, 4, 5)$ *does vanish*, so that these isolated singularities *are not* nodal points. Therefore, if a transition to another CY occurs at $\phi = \pm 1$, it is not of the usual conifold type. Note that the holomorphic change $z_1 \to e^{i\pi/4} z_1$ leaves p_1 invariant if we substitute simultaneously $\phi \to -\phi$. Therefore, from the CY point of view, ϕ and $-\phi$ parametrize the same complex structure and we could restrict to $\mathrm{Re}(\phi) \geq 0$.

However, we want to construct G_2 orbifolds of the form

$$\mathcal{G}_1 = (\mathcal{C}_1 \times S^1)/\sigma \tag{2.5}$$

with an involution $\sigma = w\mathcal{I}$, where $\mathcal{I}$ acts as an inversion on S^1 and w is antiholomorphic with fixed points on the CY ambient space

$$\sigma : \quad z_i \to \bar{z}_i \quad (i = 1, \dots, 6)\,, \qquad x^{10} \to -x^{10}\,, \tag{2.6}$$

where x^{10} is the S^1 coordinate. Clearly w commutes with the $\mathbb{C}^*$ actions. For $\mathcal{G}_1$ to be well defined, ϕ should be real so that w is a symmetry of the CY. Note that ϕ and $-\phi$ are no longer equivalent for $\mathcal{G}_1$ since $z_1 \to e^{i\pi/4} z_1$ does not commute with σ. The single family of CY's for arbitrary complex ϕ thus splits *a priori* into three G_2 branches for the orbifold parametrized by real ϕ, namely $\phi < -1$, $-1 < \phi < 1$ and $\phi > 1$.

Let us determine now the fixed point set of the orbifold. If we denote by Σ the special lagrangian 3-cycle in $\mathcal{C}_1$ fixed by w, the total fixed point set is just two copies of Σ, one at $x^{10} = 0$ and the second at $x^{10} = \pi R$, where R is the radius of S^1. A point $M = [z_1, \dots, z_6]$ then belongs to Σ if it solves (2.1) and if its equivalence class is the same as $[\bar{z}_1, \dots, \bar{z}_6]$. This means that there exist $\lambda_1, \lambda_2 \in \mathbb{C}^*$, such that $z_i = \rho_i \bar{z}_i$, $(i = 1, \dots, 6)$ with $\rho_i = \prod_{j=1}^2 \lambda_j^{Q_i^{(j)}}$, where $Q_i^{(j)}$ is the weight of z_i under $\mathbb{C}_j^*$. The scalings (2.2) and constraints (2.3) imply that $|\rho_i| = 1$, so that $\rho_i^{-1} = \bar{\rho}_i$. Therefore, if we define $z_i' = \rho_i^{-1/2} z_i = x_i + i y_i$ $(i = 1, \dots, 6)$, we can write $M = [z_1', \dots, z_6']$ with $z_i' = \bar{z}_i'$. Without loss of generality, Σ is therefore determined by the defining equation (2.1) for real unknowns x_i,

$$x_6^4 (x_1^8 + x_2^8 - 2\phi x_1^4 x_2^4) + x_3^4 + x_4^4 + x_5^4 = 0\,. \tag{2.7}$$

Notice that if x_6 could vanish in this equation, it would imply $x_{3,4,5} = 0$ as well, which is forbidden. Hence we can rescale x_6 to 1. The same is also true for x_2, since a vanishing x_2 would also imply $x_1 = 0$. The scaling actions being gauged away, eq. (2.7) is solved for x_1^4,

$$x_1^4 = \phi \pm \sqrt{\phi^2 - (1 + x_3^4 + x_4^4 + x_5^4)}\,. \tag{2.8}$$

We see that for $\phi < 1$, there is no real solution and thus Σ is empty. For $\phi = 1$, when the CY is singular, there are solutions for $x_3 = x_4 = x_5 = 0$, $x_1^4 = 1$ and Σ consists of two points

$$\Sigma = \{(1,1,0,0,0,1), (-1,1,0,0,0,1)\}\,. \tag{2.9}$$

Finally, for $\phi > 1$, we find it convenient to define the variables

$$u_1 = x_1^4 - \phi\,, \quad u_j = x_j^2\,\mathrm{sign}(x_j)\,, \qquad (j = 3,4,5)\,, \tag{2.10}$$

in which the equation for Σ reads

$$u_1^2 + u_3^2 + u_4^2 + u_5^2 = \phi^2 - 1\,, \tag{2.11}$$

describing an S^3 of radius $\sqrt{\phi^2-1}$. However, whereas $u_{3,4,5}$ and $x_{3,4,5}$ are in one-to-one correspondence, the map from x_1 to u_1 is two-to-one. Furthermore, from $-\sqrt{\phi^2-1} \le x_1^4 - \phi \le \sqrt{\phi^2-1}$ we get

$$0 < (\phi - \sqrt{\phi^2-1})^{1/4} \le x_1 \le (\phi + \sqrt{\phi^2-1})^{1/4} \tag{2.12}$$

or

$$-(\phi + \sqrt{\phi^2-1})^{1/4} \le x_1 \le -(\phi - \sqrt{\phi^2-1})^{1/4} < 0\,, \tag{2.13}$$

so that Σ actually consists of two copies of S^3 that do not intersect, one with $x_1 > 0$ and the other with $x_1 < 0$. As a result, the fixed point set Σ, according to the values of ϕ is

$$\begin{array}{lll} \phi < 1, & \Sigma = \emptyset & (\text{no fixed points})\,, \\ \phi = 1, & \Sigma = \{(\pm1,1,0,0,0,1)\}\,, & \\ \phi > 1, & \Sigma = S^3 \cup S^3 & \text{of radii } \sqrt{\phi^2-1}\,. \end{array} \tag{2.14}$$

We thus have two distinct phases as ϕ varies. Also, we see that the two special lagrangian 3-spheres present for $\phi > 1$ shrink to the isolated singularities given in eq. (2.4) for $k = 0, 2$ when $\phi \to 1$.

Spectrum for $\phi < 1$. Let us now consider M-theory compactified on the previous orbifolds. When $\phi < 1$, the compact space is smooth and the four-dimensional massless spectrum of M-theory is then determined by the Betti numbers of $\mathcal{G}_1$ [3]. The b_3 deformations of the G_2 structure Φ together with the flux of the eleven dimensional supergravity 3-form potential C on the 3-cycle homology classes give rise to b_3 complex scalars. In addition, the dimensional reduction of C on the 2-cycle homology classes provides us with b_2 vector bosons in four dimensions. Together with $\mathcal{N}=1$ superpartners from the reduction of the eleven dimensional gravitino, we have b_2 $\mathcal{N}=1$ vector multiplets and b_3 neutral chiral multiplets. In our case, a 2-cycle on $\mathcal{G}_1$ arises from a 2-cycle in $\mathcal{C}_1$ even under w, while the odd ones times S^1 give invariant 3-cycles on $\mathcal{G}_1$. Also, 3-cycles in $H_{1,2}$ ($H_{0,3}$) and $H_{2,1}$ ($H_{3,0}$) combine to give an equal number of even and odd 3-cycles. If we denote by $h_{11}^{\pm}$ the number of even and odd homology classes of 2-cycles on $\mathcal{C}_1$, then the relevant Betti numbers of the smooth G_2 orbifold $\mathcal{G}_1$ for $\phi<1$ are

$$b_2 = h_{11}^{+}\,, \quad b_3 = \frac{h_{30}+h_{03}}{2} + \frac{h_{21}+h_{12}}{2} + h_{11}^{-} = 1 + h_{12} + h_{11}^{-}\,. \tag{2.15}$$

In our case there are $h_{11}=2$ cohomology classes on $\mathcal{C}_1$. The first one is the pullback of the Kähler form J that determines the size of the ambient $\mathbb{CP}^1_{11222}$ on which w acts antiholomorphically as a symmetry, implying $J \to -J$. Similarly, the second cohomology class proportional to $dz_6 \wedge d\bar{z}_6$ is also odd under w. Therefore, $h_{11}^{+}=0$ and the Betti numbers read

$$b_2 = 0 \qquad \text{and} \qquad b_3 = 1 + 86 + 2 = 89\,.$$

The spectrum thus consists in

$$89 \text{ chiral multiplets.} \tag{2.16}$$

Spectrum for $\phi > 1$. In this case the massless spectrum decomposes into two pieces. The "untwisted sector" arises as before from the dimensional reduction of the eleven dimensional supergravity multiplet on the 2- and 3-cycles even under the orbifold involution $\sigma \equiv w\mathcal{I}$. As a result, it still consists of $b_2 = 0$ $\mathcal{N}=1$ vector multiplets and $b_3 = 89$ chiral multiplets. To this one has to add the states localized on the fixed point set $\Sigma \times \{0, \pi R\}$, i.e. 4 disconnected copies of S^3, [5]. Around each of these singular points, the geometry looks like $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^3$, where 3 directions in $\mathbb{R}^4$ correspond to the imaginary parts of local coordinates Z_j ($j=1,2,3$) that transform as $Z_j \longrightarrow \bar{Z}_j$ under w, while the fourth direction accounts for the S^1, while the factor $\mathbb{R}^3$ is the tangent plane on each S^3. As the "twisted states" are localized around each singular set S^3 and that the latter are disjoint, we can study the spectrum arising from only one of them. The total spectrum will consist of four copies of it. Since we do not take into account here the effect of an eventual generation of a superpotential from M-theory instantons [18], we can determine the desired spectrum by use of an adiabatic argument that consists in considering first the situation where S^3 is of large volume, i.e. ϕ large and positive. In that case M-theory is actually compactified down to 7 dimensions on a 4-space with an A_1 singularity. The resulting bosonic spectrum consists of a vector field in the adjoint of SU(2). The bosonic spectrum for finite ϕ is then simply obtained by dimensional reduction of this vector field on S^3 and gives an SU(2) vector boson in 4 dimensions plus $b_1(S^3) = 0$ real scalars (no Wilson lines). $\mathcal{N} = 1$ supersymmetry then assures that we have in total for $\Sigma \times \{0, \pi R\}$

$$\text{1 vector multiplet of } \mathrm{SU}(2)^4 \qquad \text{and} \qquad \text{89 neutral chiral multiplets.} \tag{2.17}$$

One can now ask if it is possible or not to desingularize the orbifold fixed points and obtain a smooth manifold of G_2 holonomy. For each S^3, this means blowing up at each point a 2-sphere in the transverse 4-space. In other words, one would glue 4 copies of Eguchi-Hanson spaces X_4 times S^3. The resulting smooth manifold would then have 4 more moduli v_n $(n = 1, \ldots, 4)$ parametrizing the volumes of the 4 blow up S^2's. However, on a G_2 manifold, there are only 3-cycle moduli and the S^2's have to combine with a 1-cycle γ_1 of radius r so that the G_2 moduli would take the form $v_n r$. Since $b_1(X_4) = 0$, γ_1 should be chosen on S^3. However, the latter has $b_1 = 0$ and the resolution is not possible. From this discussion, we see that a necessary condition for a G_2 resolution of the orbifold singularities to be possible is $b_1(\Sigma_p) \geq 1$, where $\Sigma = \cup_p^P \Sigma_p$, for connected 3-cycles Σ_p.[1] In fact, $\mathcal{N} = 1$ supersymmetry confirms this. The geometrical moduli v_n must be part of 3-cycle volumes so that they can be combined with the flux of the eleven dimensional supergravity 3-form C on the corresponding $S^2 \times \gamma_1$ to give complex scalars of chiral multiplets. Notice that the condition $b_1(\Sigma_p) \geq 1$ is the necessary one given by Joyce for a G_2 resolution to be possible and that, to be sufficient, there must be in addition a nowhere vanishing harmonic 1-form on Σ_p [19].[2] Unfortunately, this last condition happens to be difficult to check in practice [20].

We saw that the present model cannot be resolved while keeping the holonomy in G_2. However, one can still wonder whether a Ricci flat resolution is possible, such that the holonomy is a subgroup of SO(7), but not in G_2.[3] If such Ricci flat resolutions would exist, they would describe non-supersymmetric M-theory vacua. However, one can doubt that this is possible, since switching on the blow up moduli v_n would smooth the compact space, so that we pass to a Coulomb branch of $\mathcal{N} = 1$ pure $\mathrm{SU}(2)^4$ super Yang-Mills theory with spontaneous breaking of supersymmetry and we know that such a Coulomb branch does not exist.

2.2 Physical interpretation of the $\mathrm{SU}(2)^4$ phases

The geometrical G_2 moduli space parametrizes the G_2 structure Φ, (1.1), on the underlying riemannian 7-manifold. By varying this structure, we thus vary the volumes of the associative 3-cycles, which are calibrated w.r.t. Φ, i.e. their volume is given

[1]Notice that, partial resolutions should be possible when only some Σ_p's have non-trivial b_1.

[2]Actually, in [19] Σ was implicitly supposed to be connected

[3]It is known [21] that reduced holonomy G_2 for a metric on an oriented riemannian 7-manifold implies Ricci flatness.

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by the integral of the pullback of Φ. When considering M-theory on G_2 orbifolds, these real moduli are complexified by the flux of the M-theory 3-form C through the cycles and become the lowest component of chiral fields.

In the previous section we distinguished domains according to the existence or non-existence of fixed point locii of the antiholomorphic involution (2.6). When they exist, they are associative 3-cycles and via the $\mathbb{Z}_2$ singularity in their transverse space there is an SU(2) gauge group associated to each of them. The complexified (by the C-form flux) volume V_M of the associative 3-cycle becomes the complexified (by the Θ angle) gauge coupling of the corresponding SU(2) gauge group.

Let's go back now to our model based on $\mathcal{G}_1$ to be more explicit. We considered a one-dimensional subset of the geometrical G_2 moduli space parametrized by the real modulus ϕ. For $\phi > 1$ we found a fixed point locus consisting of four associative 3-spheres whose volumes vanish as we send ϕ to unity. While varying ϕ, the real volumes of these four 3-cycles behave alike. Assuming that their homology class is actually the same so that also the C-field fluxes coincide, the four associated gauge groups have one and the same complex gauge coupling

$$V_M = \frac{1}{g_{YM}^2} + i\Theta\,. \tag{2.18}$$

For $\phi < 1$ the antiholomorphic involution acts freely and there is no sign of any non-abelian gauge symmetry.

The point in the M-theory compactification is [6], that the physical moduli space is complex and that due to holomorphicity of the $\mathcal{N} = 1$ theory singularities occur at least in complex codimension one. Hence, in M-theory we can continuously deform from a theory at $\phi \gg 1$ to one at $\phi \ll -1$ without encountering any singularity. The running of the coupling suggests that the region $\phi \gg 1$ of large positive volume should correspond to the UV, where the $\mathrm{SU}(2)^4$ gauge theory is weakly coupled and nonperturbative corrections are suppressed. M-theory in the region $\phi \ll -1$ should then describe the confining phase of the same theory in the IR, with no sign of gauge symmetry.

In the noncompact model of [6] the transition between these phases could be interpreted as a flop transition between two associative 3-spheres, such that only in one of the geometric phases the orbifold group defining the theory acted freely. We would now like to see if such a flop transition takes place in our compact model as well.

The two S^3's in eq. (2.14) for $\phi > 1$ being fixed point locii of the antiholomorphic involution w acting on the whole CY ambient space, our strategy is then to consider other involutions that would fix 3-spheres for $\phi < 1$ that vanish for $\phi \to 1^-$. If we consider first diagonal involutions $z_i \to e^{i\theta_i}\bar{z}_i$ $(i = 1, \ldots, 6)$, one finds that there are actually 256 inequivalent choices of the phases θ_i consistent with the defining equation $p_1 = 0$. Among the 256 fixed point sets, 16 of them present for $\phi > 1$

vanish when $\phi \to 1^+$ and turn out to be empty for $\phi < 1$. Actually, these sets are 16 copies of $S^3 \cup S^3$. More precisely there exist 8 vanishing S^3's for each of the 4 singular points in eq. (2.4) including the ones that are fixed by the involution w we chose.

Another set of involutions consist in $z_1 \to e^{i\theta_1}\bar{z}_2$, $z_2 \to e^{i\theta_2}\bar{z}_1$, $z_j \to e^{i\theta_j}\bar{z}_j$ ($j = 3,4,5,6$), where only 128 are compatible with $p_1 = 0$. The fixed point sets of 8 of them happen to vanish at $\phi = 1$ with finite volume for $\phi < 1$ and the geometry of each one of them is determined by the equation

$$z_1^8 + \bar{z}_1^8 + 2\phi z_1^4 \bar{z}_1^4 + x_3^4 + x_4^4 + x_5^4 = 0\,, \tag{2.19}$$

where z_1 is complex and $x_{3,4,5}$ are real. Defining $z_1^4 = U_1 + iV_1$, this gives

$$2(\phi+1)U_1^2 + x_3^4 + x_4^4 + x_5^4 = 2(1-\phi)V_1^2\,, \tag{2.20}$$

in which we can rescale $V_1 = \pm 1$. We thus obtain, for each of these 8 involutions

$$\begin{array}{ll} \phi > 1, & \emptyset \quad \text{(no fixed points)}\,, \\ \phi = 1, & \{(i^k,1,0,0,0,1)\,, (k=0,1,2,3)\} \\ \phi < 1, & \bigcup_{n=1}^4 \tilde{S}^3 \quad \text{of radii } \sqrt{2(1-\phi)}\,. \end{array} \tag{2.21}$$

Therefore, there is a one-to-one correspondence between the sixteen $S^3 \cup S^3$'s at $\phi > 1$ and the eight $\bigcup_{n=1}^4 \tilde{S}^3$ at $\phi < 1$. As a consequence, on the G_2 orbifold, the two copies of $S^3 \cup S^3$ in eq. (2.14) at $x^{10} = 0$ and $x^{10} = \pi R$ give rise to 4 S^3's that undergo flop transitions.

For completeness, we note that involutions involving $z_3 \to e^{i\theta_3}\bar{z}_4$, $z_4 \to e^{i\theta_4}\bar{z}_3$, $z_5 \to e^{i\theta_5}\bar{z}_5$ and their permutations do not give rise to vanishing cycles at $\phi = 1$.

2.3 A model with conifold transition

The previous model is probably one of the simplest generalizations of the M-theory phase transition considered in [6] to the case of a compact G_2 manifold and SU(2) singularity. But still the whole M-theory discussion takes place in the moduli space of a single given G_2 space. In [4], however, the possibility of a transition into the moduli space of another smooth G_2 manifold was noticed. This happens at the conifold point of the underlying CY manifold which is a common boundary point in the complex structure moduli space of one CY family and the (complexified) Kähler moduli space of another family. From the four-dimensional point of view this corresponds to a new branch for the abelian/scalar sector of the effective field theory. Therefore, we would now like to discuss in a compact model the appearance of this additional branch via *small resolution* of the CY.

Let us choose again a one parameter sub-family of threefolds $\mathcal{C}_2$ within $\mathbb{CP}^4_{11222}[8]$, defined by

$$p_2 = z_6^4(z_1^8 + z_2^8 - 2\phi z_1^4 z_2^4) + (z_3^2 - \phi z_6^2 z_2^4)^2 + (z_4^2 - \phi z_6^2 z_2^4)^2 + (z_5^2 - \phi z_6^2 z_2^4)^2 = 0 \tag{2.22}$$

with complex parameter ϕ. The family $\mathcal{C}_2$ has singular members for $\phi = \pm 1$, $\phi = \pm i$ and $\phi = \pm i/\sqrt{2}$ where p_2 becomes non-transverse. As in section 2.1, the singular points lie in charts $z_2 \neq 0$, $z_6 \neq 0$, so that we can rescale them to 1. In anticipation of the antiholomorphic involution that we are going to take, we shall be interested in the singularities that occur for real ϕ,

$$\begin{array}{lll} \text{for } \phi = +1 : & (i^k, 1, \pm 1, \pm 1, \pm 1, 1) & , (k = 0, \dots, 3) \\ \text{for } \phi = -1 : & (i^k e^{i\pi/4}, 1, \pm 1, \pm 1, \pm 1, 1) & , (k = 0, \dots, 3) , \end{array} \tag{2.23}$$

where the $+/-$ signs are independent, giving rise to 32 singular points on the CY in each case. The reason why we added the monomials $z_A^2 z_6^2 z_2^4$, $(A = 3, 4, 5)$, is that they render the matrix of second derivatives of p_2 regular. The singularities on $\mathcal{C}_2$ are thus nodal points and ϕ sits at the conifold points in the complex structure moduli space of $\mathcal{C}_2$.

As in the previous model we proceed by restricting ϕ to real values and consider the G_2 orbifold

$$\mathcal{G}_2 = (\mathcal{C}_2 \times S^1)/\sigma \,, \tag{2.24}$$

where $\sigma = w\mathcal{I}$ is defined as in eq. (2.6). The special lagrangian 3-cycle Σ of w-invariant points in $\mathcal{C}_2$, giving the σ-fixed points $\Sigma \times \{0, R\pi\}$, is determined by

$$x_6^4(x_1^8 + x_2^8 - 2\phi x_1^4 x_2^4) + (x_3^2 - \phi x_6^2 x_2^4)^2 + (x_4^2 - \phi x_6^2 x_2^4)^2 + (x_5^2 - \phi x_6^2 x_2^4)^2 = 0 \,, \tag{2.25}$$

where the unknowns are real and x_2, x_6 can be scaled to 1, thanks to eq. (2.3). Solving for x_1^4, one obtains

$$x_1^4 = \phi \pm \sqrt{\phi^2 - [1 + (x_3^2 - \phi)^2 + (x_4^2 - \phi)^2 + (x_5^2 - \phi)^2]} \,, \tag{2.26}$$

which implies $\phi \geq 1$. In the variables

$$u_1 = x_1^4 - \phi \ , \quad u_j = x_j^2 - \phi \,, \quad (j = 3, 4, 5) \,, \tag{2.27}$$

we find again a 3-sphere

$$u_1^2 + u_3^2 + u_4^2 + u_5^2 = \phi^2 - 1 \,. \tag{2.28}$$

Now, the four maps (2.27) from x_i to u_i are two-to-one and by the same reasoning as in the previous model one easily shows that (for finite ϕ) the x's are either positive definite or negative definite. Hence equation (2.28) gives rise to a total of 16 disjoint copies of S^3. To summerize the situation, we have two distinct phases along the real ϕ axis,

$$\begin{array}{ll} \phi < 1, & \Sigma = \emptyset \quad \text{(no fixed points)} \,, \\ \phi = 1, & \Sigma = \{(\pm 1, 1, \pm 1, \pm 1, \pm 1, 1)\} \quad \text{(i.e. 16 points)} \,, \\ \phi > 1, & \Sigma = \bigcup_{n=1}^{16} S^3 \quad \text{of radii } \sqrt{\phi^2 - 1} \,, \end{array}$$

where the w-invariant S^3's that shrink to 16 of the 32 nodes in eq. (2.23) when $\phi \to 1^+$ are precisely the ones involved in the conifold transitions of the underlying CY $\mathcal{C}_2$. In the next section we shall discuss the new branches of G_2 manifolds emanating from $\phi = \pm 1$ via these CY conifold transitions.

Spectrum. Along the ϕ branches, the present orbifold $\mathcal{G}_2$ is similar to $\mathcal{G}_1$. For completeness, we recall that the massless spectrum describes $b_3 = 89$ neutral chiral multiplets present for arbitrary ϕ from the untwisted sector together with a pure SU(2) super Yang-Mills theory in non-abelian ($\phi \gg 1$) or confining ($\phi \ll -1$) phase for each of the 32 S^3 components of $\Sigma \times \{0, \pi R\}$. In total we have

$$1 \text{ vector multiplet of } \mathrm{SU}(2)^{32} \qquad \text{and} \qquad 89 \text{ neutral chiral multiplets}\,. \tag{2.29}$$

2.4 The extremal transition

Following [9], we would like to consider in this section the conifold transition that occurs in $\mathcal{C}_2$ at $\phi = \epsilon \equiv \pm 1$. We shall then generalize the results of [4] for the descendant extremal transitions between G_2 spaces.

At $\phi = \epsilon$, the non-transverse defining equation (2.22) of $\mathcal{C}_2$ takes the form

$$p_{\phi=\epsilon} = \mathrm{Det}\begin{pmatrix} P_{11}(z) & P_{12}(z) \\ -\bar{P}_{12}(z) & \bar{P}_{11}(z) \end{pmatrix}, \tag{2.30}$$

where

$$P_{11}(z) = z_6^2(z_1^4 - \epsilon z_2^4) + i(z_3^2 - \epsilon z_6^2 z_2^4)\,, \qquad P_{12}(z) = (z_4^2 - \epsilon z_6^2 z_2^4) - i(z_5^2 - \epsilon z_6^2 z_2^4)\,, \tag{2.31}$$

and $\bar{P}_{11}(z)$, $\bar{P}_{12}(z)$ are obtained by simply changing i to $-i$ in the coefficients. The so called *small resolution* of this singular variety consists in replacing the 32 nodes given in eq. (2.23) by 2-spheres S^2. The result can then be written as an intersection of hypersurfaces defined by

$$\begin{cases} P_{11}(z)t_1 + P_{12}(z)t_2 = 0 \\ -\bar{P}_{12}(z)t_1 + \bar{P}_{11}(z)t_2 = 0\,, \end{cases} \tag{2.32}$$

where (t_1, t_2) are projective coordinates parametrizing a $\mathbb{CP}^1 \equiv S^2$. In fact, since $(t_1, t_2) \neq (0, 0)$, the determinant of the coefficients of $t_{1,2}$, which is precisely $p_{\phi=\epsilon}$ in eq. (2.30), must vanish and the system in eq. (2.32) is equivalent to

$$\begin{cases} P_{11}(z)t_1 + P_{12}(z)t_2 = 0 \\ p_{\phi=\epsilon} = 0\,, \end{cases} \tag{2.33}$$

where the first equation determines (t_1, t_2) in $\mathbb{CP}^1$. When the volume of $\mathbb{CP}^1$ is sent to zero, the first equation is irrelevant and we recover the singular CY, while when

$\mathbb{CP}^1$ is of finite volume, the manifold $\mathcal{C}_2^\epsilon$ we obtain is smooth and sits at a generic point of the family of complete intersection CY's

$$\begin{bmatrix} \mathbb{CP}^4_{11222} & \| & 4 \; 4 \\ \mathbb{CP}^1 & \| & 1 \; 1 \end{bmatrix}, \tag{2.34}$$

whose Hodge numbers are $h'_{11} = 3$ and $h'_{12} = 55$.[4]

We would like to see now if we can construct G_2 manifolds out of these new CY branches. We therefore consider products $\mathcal{C}_2^\epsilon \times S^1$ and look for an antiholomorphic involution w' acting on the whole $\mathcal{C}_2^\epsilon$ ambient space $\mathbb{CP}^4_{11222} \times \mathbb{CP}^1$. Since we know that z_i should be sent to $\bar{z}_i$, we need to extend the antiholomorphic action on the coordinates (t_1, t_2) of $\mathbb{CP}^1$. This action must be linear in order to preserve the Kähler metric of $\mathbb{CP}^1$, so that there must be a 2×2 matrix M such that

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \to M \begin{pmatrix} \bar{t}_1 \\ \bar{t}_2 \end{pmatrix}, \qquad MM^* = \lambda I\,, \qquad MM^\dagger = \mu I\,, \tag{2.35}$$

where the two conditions on M assure that the transformation is of order two and preserves the Kähler metric of $\mathbb{CP}^1$. As a result, the most general matrix M (whose determinant can be normalized without loss of generality) is

$$\begin{pmatrix} a & iB \\ iB & \bar{a} \end{pmatrix} \quad \text{where } \; |a|^2 + B^2 = 1\,,\; a \in \mathbb{C}\,,\; B \in \mathbb{R} \quad \text{or} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.36}$$

The first transformations always have fixed points in $\mathbb{CP}^1$ (as can be seen by diagonalizing them, see [4] for details), while the second is freely acting (since, otherwise, $t_1 = t_2 = 0$, which is forbidden). Actually, among all these involutions, only the freely acting one can be combined such that

$$w' : \quad z_i \to \bar{z}_i \quad (i = 1, \ldots, 6)\,, \qquad t_1 \to -\bar{t}_2\,,\; t_2 \to \bar{t}_1\,, \tag{2.37}$$

is a symmetry of $\mathcal{C}_2^\epsilon$, as can be seen from

$$\begin{cases} P_{11}(z)y_1 + P_{12}(z)y_2 = 0 \\ -\bar{P}_{12}(z)y_1 + \bar{P}_{11}(z)y_2 = 0 \end{cases} \xrightarrow{w'} \begin{cases} -P_{11}(\bar{z})\bar{y}_2 + P_{12}(\bar{z})\bar{y}_1 = 0 \\ \bar{P}_{12}(\bar{z})\bar{y}_2 + \bar{P}_{11}(\bar{z})\bar{y}_1 = 0\,, \end{cases} \tag{2.38}$$

after complex conjugation. We thus obtain two new branches of smooth G_2 manifolds $(\mathcal{C}_2^\epsilon \times S^1)/w'\mathcal{I}$ connected at $\phi = \pm 1$ to the original ones $(\mathcal{C}_2 \times S^1)/w\mathcal{I}$.

Actually, as soon as we see that such branches exist, it is not a surprise to observe that the orbifolds $(\mathcal{C}_2^\epsilon \times S^1)/w'\mathcal{I}$ cannot have fixed points. In fact, such a fixed point set would have been composed of two copies of a special lagrangian 3-cycle Σ^ϵ on $\mathcal{C}_2^\epsilon$.

[4] These Hodge numbers can be determined as follows. The Lefschetz hyperplane theorem implies that h'_{11} is given by the dimension of the Kähler moduli space of the ambient space, therefore $h'_{11} = 2 + 1$. Then, h'_{12} is deduced from the relation $\chi' - \chi = 2N$, where χ (χ') is the Euler characteristic of $\mathcal{C}_2$ ($\mathcal{C}_2^\epsilon$) and $N = 32$ is the number of nodes.

Since on the G_2 branches built from $\mathcal{C}_2^\epsilon$ we vary Kähler classes of the CY, $\mathrm{vol}(\mathbb{CP}^1)$ in our present model, the 3-cycle volume $\mathrm{vol}(\Sigma^\epsilon)$ would remain constant when moving on this branch. As a result, by sending back $\mathrm{vol}(\mathbb{CP}^1) \to 0$ towards the conical CY where $\Sigma^\epsilon = \Sigma$, we would find $\mathrm{vol}(\Sigma^\epsilon) = 0$. However, since $\mathcal{C}_2^\epsilon$ is not singular, this implies that Σ^ϵ is empty. In fact, for $\phi = -1$, this was ensured by the fact that $p_{\phi=-1}$ in eq. (2.33) has no solution when restricted to real unknowns x_i's. However, at $\phi = +1$, since $p_{\phi=+1}$ admits real solutions, the antiholomorphic involution w' has to act as $t_1 \to -\bar{t}_2$, $t_2 \to \bar{t}_1$, so that it is freely acting on $\mathcal{C}_2^{+1}$.

Spectrum on $(\mathcal{C}_2^\epsilon \times S^1)/w'\mathcal{I}$**.** The spectrum on theses branches is then immediately found. Since $(dT \wedge d\bar{T})/|T|^2$, where $T = t_1/t_2$ is odd under w', the third 2-cycle homology class of $\mathcal{C}_2^\epsilon$ is odd (together with the two odd classes already present on $\mathcal{C}_2$). Consequently, the Betti numbers take the values

$$b_2' = h_{11}^{+'} = 0\,, \qquad b_3' = 1 + h_{12}' + h_{11}^{-'} = 59\,, \tag{2.39}$$

and the massless spectrum consists of 59 chiral multiplets.

2.5 Physical interpretation of the new phases

We would now like to interpret the phases described in the previous section from a physical point of view. This means that instead of considering geometrical moduli spaces of real dimensions b_3 (b_3'), we have instead to interpret the branches complexified with the eleven dimensional supergravity 3-form C. The physical moduli space coordinates are then the scalar components of the massless chiral fields.

The extremal transition at $\phi = -1$ in characterized by 32 nodal points on the underlying CY that are not fixed under the $\mathbb{Z}_2$ involution. These nodes are vanishing 2-spheres on $\mathcal{C}_2^{-1}$ whose classes are odd under w' (and equal in this example). This situation, where b_3 varies and b_2 is constant, is precisely one considered in [4] where it was shown how this describes a change of branch in the scalar potential, the abelian gauge group not being affected. At $\phi = +1$, we saw that 16 of the nodes are mapped into each other under $\mathbb{Z}_2$, while the 16 others are invariant. Again, the single vanishing class of these nodes is odd. Hence this situation, where fixed nodes arise, is a generalization of the previous case we discuss now.

Following [10, 11], suppose we have a CY $\mathcal{C}'$ where N 2-cycles γ_a ($a = 1, \ldots, N$) shrink to zero size at some point in the Kähler moduli space. A priori not linearly independent in homology, their classes satisfy R relations

$$\alpha_1^r[\gamma_1] + \cdots + \alpha_N^r[\gamma_N] = 0 \qquad (r = 1, \ldots, R)\,, \tag{2.40}$$

with integer coefficients. When M-theory is compactified on $\mathcal{C}' \times S^1$, membranes wrapped on these cycles give N black hole hypermultiplets charged under the $(N-R)$ independent U(1)'s associated with the vanishing classes. In our case, $N = 32$ and

$N-R=1$. If we denote by $(h_a, \tilde{h}_a)$ $(a=1,\ldots,N)$ and T^I $(I=1,\ldots,N-R)$ the complex scalars of the hypermultiplets and vector multiplets, respectively, $\mathcal{N}=2$ supersymmetry in four dimensions implies the presence of the superpotential

$$\mathcal{W} = \sum_{I=1}^{N-R} \sum_{a=1}^{N} q_I^a T^I h_a \tilde{h}_a \,, \tag{2.41}$$

where q_I^a is the charge of $(h_a, \tilde{h}_a)$ under the I-th U(1). Since the antiholomorphic involution is an isometry, it maps the set of nodes into itself. We can therefore define $N = N' + N''$, where N' is even and counts the number of non-invariant nodes, while N'' is the number of fixed nodes. We order the nodal points so that the vanishing 2-cycles on $\mathcal{C}'$ satisfy $\gamma_b \to -\gamma_{b+N'/2}$ $(b=1,\ldots,N'/2)$ and $\gamma_c \to -\gamma_c$ $(c = N'+1,\ldots,N'+N'')$, the minus sign being a consequence of the fact that these cycles are calibrated and thus holomorphic while the involution is antiholomorphic. In these conventions we have

$$\begin{aligned}
h_b(z,T,x^{10}) &= \tilde{h}_{b+N'/2}\left(\bar{z}, -\frac{1}{\overline{T}}, -x^{10}\right), \\
\tilde{h}_b(z,T,x^{10}) &= h_{b+N'/2}\left(\bar{z}, -\frac{1}{\overline{T}}, -x^{10}\right), \\
h_c(z,T,x^{10}) &= \tilde{h}_c\left(\bar{z}, -\frac{1}{\overline{T}}, -x^{10}\right),
\end{aligned} \tag{2.42}$$

so that the sums $H_b = h_b + \tilde{h}_{b+N'/2}$, $\tilde{H}_b = \tilde{h}_b + h_{b+N'/2}$ and $H_c = h_c + \tilde{h}_c$ are even under the involution $\sigma' = w'\mathcal{I}$ and become the scalar components of $\mathcal{N}=1$ chiral multiplets, while the differences are projected out. The superpotential (2.41) now becomes

$$\mathcal{W} = \sum_{I=1}^{N-R} \left(\sum_{b=1}^{N'/2} q_I^b T^I H_b \tilde{H}_b + \frac{1}{2} \sum_{c=N'+1}^{N'+N''} q_I^c T^I H_c H_c \right), \tag{2.43}$$

with classical diagonal kinetic terms. The scalar potential has different phases. When the scalars T^I have non-vanishing vacuum expectation values, all the black hole chiral multiplets are massive. This branch describes the model on the manifold $(\mathcal{C}' \times S^1)/w'\mathcal{I}$ (with $b_3' = 59$ chiral multiplets in our example). In the reverse situation where the black holes condense, the T^I fields become massive. The massless spectrum contains in addition to the b_3' chiral multiplets the $N'/2$ pairs $(H_b, \tilde{H}_b)$ and N'' fields H_c, from which we substract $(N-R)$ complex degrees of freedom arising from the vanishing F-term conditions associated to the T^I's and another $(N-R)$ from the massive T^I's themselves. On this branch, the massless spectrum is then composed of an equal number b_2 (equal to 0 in our example) of U(1) factors and b_3 chiral multiplets given by[5]

$$b_2 = b'_2 \qquad \text{and} \qquad b_3 = b'_3 + 2R - N\,, \tag{2.44}$$

in concordance with the model of the present section, $b_3 = 59 + 2 \cdot 31 - 32 = 89$ for $\phi = 1$ ($\phi = -1$), where $N' = 16$, $N'' = 16$, ($N' = 32$, $N'' = 0$), respectively. We see that actually only the total number of nodes N is relevant in the transition. However, the pairs of chiral fields associated to H_b and $\tilde{H}_b$ form an $\mathcal{N} = 2$ sector in the theory, while the fields H_c are truly part of $\mathcal{N} = 1$ multiplets. From an M-theory point of view, the former are associated with membranes wrapped on copies of $\mathbb{CP}^1$ identified two by two on the orbifold, so that they are unoriented, while the latter are associated with unoriented membranes wrapped on $\mathbb{RP}^2$.

Having extended the validity of eq. (2.44) for extremal transitions at constant b_2 to the case where nodal points are fixed under the involution, we would like to note that there is no similar extension for the case where b_2 varies and describes a Higgs mechanism of the abelian gauge group. The reason is that the class of a fixed nodal point satisfies $[\gamma_c] \to -[\gamma_c]$ under the antiholomorphic involution and that the Higgs effect described in [4] arises when a vanishing homology class is even.

3. Models with adjoint matter

3.1 A model based on $(\mathbb{CP}^4_{11222}[8] \times S^1)/\mathbb{Z}_2$

We have seen before that the compactification of M-theory on G_2-holonomy spaces with associative cycles Σ of A_{N-1} singularities describes SU(N) gauge theories with $b_1(\Sigma)$ Wilson lines in chiral multiplets. In section 2, we presented models with $b_1(\Sigma) = 0$. Here, we will discuss a simple realization of a gauge theory with matter in the adjoint arising from $b_1(\Sigma) > 0$. For this, let us consider another family of CY's $\mathcal{C}_3$ in $\mathbb{CP}^4_{11222}$, similar to the one considered in section 2.1. The Hodge numbers are still $h_{11} = 2$ and $h_{12} = 86$ and the defining polynomial is now

$$p_3 = z_6^4(z_1^8 + z_2^8) + z_3^4 + z_4^4 + z_5^4 - 2\theta z_4^2 z_5^2 = 0\,, \tag{3.1}$$

where θ is a complex parameter. Thanks to eq. (2.3), $\mathcal{C}_3$ happens to be singular when $z_3 = z_6 = 0$, $\theta = \pm 1$ and $z_4^2 = \theta z_5^2 \neq 0$. As a result, we can set z_5 to 1 by a $\mathbb{C}_1^*$ rescaling and obtain the sets of singular points,

$$\begin{aligned} &\text{for } \theta = +1: \quad (z_1, z_2, 0, \pm 1, 1, 0)\,, \\ &\text{for } \theta = -1: \quad (z_1, z_2, 0, \pm i, 1, 0)\,, \end{aligned} \tag{3.2}$$

[5] We hope the reader will not be confused by the fact that we reversed the notations of ref. [4] for the primed and not primed Betti numbers.

where $(z_1, z_2) \neq (0,0)$ are arbitrary and modded out by the $\mathbb{C}_2^*$ action, so that they parametrize a $\mathbb{CP}^1$. As a consequence, two non-intersecting genus 0 curves of singularities occur on $\mathcal{C}_3$ when $\theta = \pm 1$.

To construct G_2 manifolds out of this family of CY's, we consider the involution $\sigma = w\mathcal{I}$ in eq. (2.6) and define

$$\mathcal{G}_3 = (\mathcal{C}_3 \times S^1)/\sigma\,, \tag{3.3}$$

where the complex structure parameter θ is now restricted to be real for w to be an isometry of $\mathcal{C}_3$. Before complexification with the M-theory 3-form C, the G_2 moduli space is again split in three disconnected pieces, $\theta < 1$, $-1 < \theta < 1$ and $\theta > 1$.

The fixed point set on $\mathcal{G}_3$ is again two copies at $x^{10} = 0$ and $x^{10} = \pi R$ of the special lagrangian 3-cycle Σ fixed by w on $\mathcal{C}_3$. Its equation reduces to the polynomial p_3 for real unknown x_i,

$$x_6^4(x_1^8 + x_2^8) + x_3^4 + x_4^4 + x_5^4 - 2\theta x_4^2 x_5^2 = 0\,. \tag{3.4}$$

From eq. (2.3), the factor multiplying x_6^4 is always strictly positive. Therefore, $x_5 = 0$ would also imply $x_3 = x_4 = x_6 = 0$ which is forbidden, so that we can rescale x_5 to 1. Now, we observe that $(x_1, x_2) \neq (0,0)$ modded out by the real rescaling reminiscent to $\mathbb{C}_2^*$ parametrize a full $\mathbb{RP}^1 \equiv S^1$. Let us now fix a point (x_1, x_2) in this S^1, and then define the one-to-one change of variables

$$u_6 = x_6^2 \sqrt{x_1^8 + x_2^8}\ \mathrm{sign}(x_6)\,, \quad u_3 = x_3^2\ \mathrm{sign}(x_3)\,, \tag{3.5}$$

and write eq. (3.4) as

$$u_6^2 + u_3^2 = -(x_4^4 - 2\theta x_4^2 + 1)\,. \tag{3.6}$$

This equation has solutions when the r.h.s. is positive i.e. when

$$0 < (\theta - \sqrt{\theta^2 - 1})^{1/2} \leq x_4 \leq (\theta + \sqrt{\theta^2 - 1})^{1/2} \tag{3.7}$$

or

$$-(\theta + \sqrt{\theta^2 - 1})^{1/2} \leq x_4 \leq -(\theta - \sqrt{\theta^2 - 1})^{1/2} < 0\,, \tag{3.8}$$

implying $\theta \geq 1$. As a conclusion, when choosing x_4 in these segments, eq. (3.6) describes a circle parametrized by $u_{6,3}$ whose radius vanishes at the boundaries of the x_4 intervals. We thus obtain two 2-spheres at each point of the S^1 parametrized by $x_{1,2}$. Since the S^2's are disjoint, this results in two copies of $S^2 \times S^1$. To summerize, the fixed point set according to the values of θ is

$$\begin{array}{ll}
\theta < 1, & \Sigma = \emptyset \quad \text{(no fixed points)}\,, \\
\theta = 1, & \Sigma = \{(x_1, x_2, 0, \pm 1, 1, 0)\} \quad \text{i.e.} \quad S^1 \cup S^1\,, \\
\theta > 1, & \Sigma = (S^2 \times S^1) \cup (S^2 \times S^1)\,,
\end{array}$$

where the radius of the S^2's is of order $\sqrt{\theta - 1}$ when $\theta \to 1^+$. Note that the complex deformation of $\mathcal{C}_3$ can determine the volume of these 2-spheres since they are not holomorphic on the CY. Also, at $\theta = 1$, the S^1's are the equators of the 2-spheres of singularities in eq. (3.2). We thus have a transition between two distinct phases $\theta < 1$ and $\theta > 1$. As in section 2.4, there could also be extremal transitions at $\theta = \pm 1$ giving rise to new branches of G_2 manifolds. However, we don't consider this situation in this work.

Spectrum for $\theta < 1$. On this branch, the orbifold is smooth and the spectrum is thus simply determined by the Betti numbers from eq. (2.15) to give the same result we had for $\mathcal{G}_1$ when $\phi < 1$,

$$b_2 = 0 \qquad \text{and} \qquad b_3 = 1 + 86 + 2 = 89\,,$$

i.e. 89 chiral multiplets with no gauge group.

Spectrum for $\theta > 1$. In addition to the previous spectrum, one now has to take into account the states that sit at each connected component of the fixed point set $\Sigma \times \{0, \pi R\}$. As in the case $\phi > 1$ for $\mathcal{G}_1$, for each $\Sigma_p = S^2 \times S^1$ ($p = 1, 2$), they fill a $\mathcal{N} = 1$ multiplet in 7 dimensions in the adjoint of SU(2) compactified down to 4 dimensions on Σ_p giving rise to a $\mathcal{N} = 1$ vector multiplet. In addition, we get in general $b_1(\Sigma_p)$ (equal to 1 in our case) real scalars in the adjoint of SU(2), arising from the flux of the 7 dimensional gauge boson on the 1-cycles of Σ_p. Notice that the vacuum expectation value of these scalars are zero since otherwise they would break the gauge group. Actually, they are combined with the volume of the vanishing 2-sphere at the A_1 singularity times the radius r_α ($\alpha = 1, \ldots, b_1(\Sigma_p)$) to give $b_1(\Sigma_p)$ complex scalars in the adjoint of SU(2). As a result, the total spectrum in the present case consists in

$$\begin{gathered}\text{1 vector and 1 chiral multiplet in the adjoint of } \mathrm{SU}(2)^4 \\ \text{and 89 neutral chiral multiplets}\,. \end{gathered} \tag{3.9}$$

Since $b_1(\Sigma_p) > 0$, we can consider now the desingularization of $\mathcal{G}_3$ while keeping the G_2 holonomy. To this end we have to blow up the 2-spheres vanishing at each A_1 singularity. This amounts to give a mass to the membranes wrapped on them and corresponds physically to move in the Coulomb branch $\mathrm{SU}(2)^4 \to \mathrm{U}(1)^4$. The chiral multiplets associated to these 2-cycles get a vacuum expectation value and become massive. As a result, in addition to the "untwisted" $b_2 = 0$ vector multiplets and $b_3 = 89$ chiral multiplets, we now have a "twisted sector" consisting of $2P$ vector multiplets and $2\sum_{p=1}^{P} b_1(\Sigma_p)$ neutral chiral multiplets (where $P = 2$ in the present model). In general, in the Coulomb branch of each of the $2P$ SU(2) coupled to $b_1(\Sigma_p)$ chiral multiplets, we have

$$b_2 = h_{11}^{+} + 2P \qquad \text{vector multiplets} \tag{3.10}$$

and

$$b_3 = 1 + h_{12} + h_{11}^- + 2\sum_{p=1}^{P} b_1(\Sigma_p) \qquad \text{chiral multiplets}\,. \tag{3.11}$$

Note that this result was conjectured in [15] for the case $P = 1$ by considering the dual description in type IIA in the presence of D6-branes and O6-planes.[6] In our example, since we have $P = 2$ disconnected components $S^2 \times S^1$ in Σ, each of them with $b_1 = 1$ the full massless theory turns out to be a $\mathrm{U}(1)^4$ gauge theory and $89 + 4 = 93$ neutral chiral multiplets. The formulas (3.10) and (3.11) are actually an extension of the case considered by Joyce in the desingularization of orbifolds T^7/Γ, where the fixed points are $2P$ copies of non-intersecting T^3's [8, 19].

Finally, let us note that in this example, the twisted sector realizes a classical pure $\mathcal{N} = 2$ SU(2) super Yang-Mills theory in M-theory, however coupled to the $\mathcal{N} = 1$ chiral multiplet associated to θ.

3.2 Yet another model with $S^2 \times S^1$ flops

There are other CY's considered in the literature containing special lagrangian 3-cycles with $b_1(\Sigma) > 0$. As an example, model III of [22] is defined by the polynomial

$$p_4 = z_6^4(z_1^8 + z_2^8 - 2\phi z_1^4 z_2^4) + z_3^4 + z_7^2(z_4^4 + z_5^4) = 0\,, \tag{3.12}$$

in $\mathbb{CP}^4_{11222}$, where the z_7 coordinate arises by orbifolding our model $\mathcal{C}_1$ by the $\mathbb{Z}_2$ action $(z_4, z_5) \to (-z_4, -z_5)$ and blowing up the singularity. Therefore, there is a third $\mathbb{C}^*$ action acting as $(z_4, z_5, z_7) \to (\lambda z_4, \lambda z_5, \lambda^{-2} z_7)$ and the excluded set of the ambient toric variety is $(z_1, z_2) \neq (0, 0)$, $(z_4, z_5) \neq (0, 0)$ and $(z_3, z_6) \neq (0, 0)$. The Hodge numbers of the CY are $h_{11} = 3$ and $h_{12} = 55$. With the involution (2.6) extended by $z_7 \to \bar{z}_7$, the analysis of this model is analogous to the model based on $\mathcal{C}_1$. The spectrum for $\phi < 1$ turns out to be

$$b_2 = 0 \qquad \text{and} \qquad b_3 = 1 + 55 + 3 = 59\,,$$

i.e. 59 chiral multiplets with no gauge group. For $\phi > 1$, eq. (2.11) gets modified to

$$u_1^2 + u_3^2 + u_7^2 = \phi^2 - 1\,, \tag{3.13}$$

where $u_7 = x_7\sqrt{x_4^4 + x_5^4}$. Here, (x_4, x_5) parametrize an $\mathbb{RP}^1 \equiv S^1$ moded out by $(z_4, z_5) \to (-z_4, -z_5)$ giving rise to another S^1 of half radius, and eq. (3.13) defines an S^2 in u-variables. Therefore, the S^3's of $\mathcal{C}_1$ are replaced by $S^2 \times S^1$'s in this case, so that $\Sigma = S^2 \times S^1 \cup S^2 \times S^1$ has $P = 2$ components $\Sigma_p = S^2 \times S^1$ with $b_1(\Sigma_p) = 1$. As in the previous section, the full spectrum in this branch is

$$\begin{aligned} &1 \text{ vector and 1 chiral multiplets in the adjoint of } \mathrm{SU}(2)^4 \\ &\qquad \text{and 59 neutral chiral multiplets}\,. \end{aligned} \tag{3.14}$$

[6]Eqs. (3.10) and (3.11) were also known to the authors of [4].

In addition, the $\tilde{S}^3$'s described in eq. (2.20) for $\phi < 1$ become similarly $\tilde{S}^2 \times S^1$'s given by

$$2(\phi + 1)U_1^2 + u_3^2 + u_7^2 = 2(1 - \phi)V_1^2 \,. \tag{3.15}$$

As a result, this should describe $S^2 \times S^1$ flops on the G_2 orbifold, where only the 2-spheres vanish.

Finally, the Coulomb branch occurs as before when the orbifold fixed points for $\phi > 1$ are desingularized. The resulting spectrum is then $b_2 = 0 + 4$ vector multiplets and $b_3 = 59 + 4 = 63$ chiral multiplets.

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